Finite time ruin probability with heavy-tailed insurance and financial risks

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Abstract

This paper studies the probability of ruin within a finite time for a discrete-time model, in which the insurance risk is assumed to be heavy tailed. A precise asymptotic estimate for the finite-time ruin probability is established as the initial capital increases, extending the corresponding result of Tang and Tsitsashvili [2003. Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. Stochastic Process. Appl. 108, 299–325] to the subexponential case.

Keywords: Subexponentiality; Independent product; Ruin probability; Financial risk; Insurance risk

1. Introduction

Properties of ruin probability on a stochastic economic environment have been investigated by many researchers, such as Norberg (1999), Nyrhinen (1999, 2001), Kalashnikov and Norberg (2002), Tang and Tsitsashvili (2003, 2004) and so on.

Following the work of Nyrhinen (1999, 2001) and Tang and Tsitsashvili (2003, 2004), we consider the ruin probability on a stochastic environment. Let a random variable (r.v.) $X_n$ be the net loss—the total claim amount premium minus the total incoming—of the insurer at time $n$, and a positive r.v. $Y_n$ be the discount factor from time $n$ to time $n-1$, $n = 1, 2, \ldots$. Hence the discounted surplus of the insurance company accumulated till the end of time $n$ can be modelled by a discrete time stochastic process

$$ S_0 = x, \quad S_n = x - \sum_{i=1}^{n} X_i \prod_{j=1}^{i} Y_j, \quad n = 1, 2, \ldots, $$

where $x$ is the initial capital of the insurance company. The probability of ruin within finite time is defined by

$$ \psi(x, n) = P\left( \min_{0 \leq k \leq n} S_k < 0 \right) = P(U_n > x), \quad n = 1, 2, \ldots, \quad (1.1) $$

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where \( U_n = \max\{0, \max_{1 \leq k \leq n} \sum_{i=1}^{k} X_i \prod_{j=1}^{k} Y_j \} \) with \( U_0 = 0 \). Recently, the ruin probabilities in (1.1) have been investigated by Nyrhinen (1990, 2002) and Tang and Tsitsashvili (2003, 2004), among others.

The standing assumption of the paper is that the distribution of the net loss \( X_n \) is heavy-tailed. The most important class of heavy-tailed distribution is the subexponential class. By definition, a distribution function (d.f.) \( F \) supported on \([0, \infty)\) is said to be subexponential, denoted by \( F \in \mathcal{S} \), if

\[
\lim_{x \to \infty} \frac{F^{\ast 2}(x)}{F(x)} = 2,
\]

where \( F^{\ast 2} \) denotes the 2-fold convolution of \( F \). More generally, a d.f. \( F \) supported on \((-\infty, \infty)\) is still said to be subexponential if \( F(x)\mathbb{1}_{[0, \infty)} \) is subexponential, where \( \mathbb{1}_{A} \) denotes the indicator function of \( A \). We also introduce some closely related heavy-tailed classes, a d.f. \( F \) supported on \((-\infty, \infty)\) is said to be long-tailed, denoted by \( F \in \mathcal{L} \), if

\[
\lim_{x \to \infty} \frac{F(x+1)}{F(x)} = 1;
\]
a d.f. \( F \) supported on \((-\infty, \infty)\) is said to be dominatedly varying tailed, denoted by \( F \in \mathcal{D} \), if

\[
\lim_{x \to \infty} \frac{F(\theta x)}{F(x)} < \infty
\]

for some (or, equivalently for all) \( \theta \in (0, 1) \). For more details of heavy-tailed distributions, see Embrechts et al. (1997).

Following Nyrhinen (1999, 2001), Tang and Tsitsashvili (2003), etc., we also need some general assumptions:

\begin{itemize}
  \item \( P_1 \). The successive net losses \( X_n, n = 1, 2, \ldots \), constitute a sequence of independent and identically distributed (i.i.d.) r.v.’s with generic r.v. \( X \) and common d.f. \( F \) concentrated on \((-\infty, \infty)\);
  \item \( P_2 \). The reserve is currently invested into a risky asset which may earn negative interest \( r_n \) at year \( n \), and the discount factor \( Y_n = (1 + r_n)^{-1}, n = 1, 2, \ldots \), also constitute a sequence of i.i.d. r.v.’s, with generic r.v. \( Y \) and common d.f. \( G \) concentrated on \((0, \infty)\);
  \item \( P_3 \). The two sequences \( \{X_n : n = 1, 2, \ldots \} \) and \( \{Y_n : n = 1, 2, \ldots \} \) are mutually independent.
\end{itemize}

In the terminology of Norberg (1999), we call the generic r.v.’s \( X \) and \( Y \) as the insurance risk and financial risk, respectively.

Here and throughout, all limiting relationships are for \( x \to \infty \) unless otherwise stated. For two positive functions \( a(\cdot) \) and \( b(\cdot) \), we symbolize their relation by \( a(x) \sim b(x) \) if they satisfy \( \lim_{x \to \infty} a(x)/b(x) = 1 \), and \( a(x) = \mathcal{O}(b(x)) \) if \( \limsup_{x \to \infty} a(x)/b(x) < \infty \).

One of most important results of Tang and Tsitsashvili (2003) is as follows:

**Theorem A** (Tang and Tsitsashvili (2003)). Suppose that the assumptions \( P_1, P_2 \) and \( P_3 \) hold simultaneously. If

\begin{enumerate}
  \item \( F \in \mathcal{L} \cap \mathcal{D}, \)
  \item \( \mathbb{E}Y^p < \infty \) for some large \( p > 0, \)
\end{enumerate}

then it holds for each \( n = 1, 2, \ldots \) that

\[
\psi(x, n) \sim \sum_{k=1}^{n} P \left( X \prod_{i=1}^{k} Y_i > x \right).
\]

It is well-known that

\( \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \)

(see Embrechts et al., 1997) is a proper inclusion, many important heavy-tailed distributions do not belong to \( \mathcal{L} \cap \mathcal{D} \), such as Lognorm distribution and Weibull distribution and so on. It is natural to investigate whether
(1.2) still holds if the condition $F \in \mathcal{L} \cap \mathcal{D}$ in Theorem A is extended to $F \in \mathcal{S}$. And Theorem A only consider the case that the tail of the insurance risk $X$ is heavier than that of the financial risk $Y$, then it is of interest to investigate whether (1.2) still holds if $Y$ is heavier then $X$. Unfortunately, Theorem A does not answer this question.

In this paper, we restrict ourselves to the case that the insurance risk $X$ has an absolutely continuous subexponential distribution. We derive the precise asymptotic estimates for the finite ruin probability $\psi(x, n)$ under a very relax condition on $G$. In particular, the result holds for the case even if the tail of the financial risk $Y$ is heavier than that of the insurance risk $X$. Our main results extend Theorem A to the whole subexponential class.

We will give our main results in Section 2. We noted that subexponentiality of independent product tails is the key to the proof of Theorem 2.1, thus we will first cite our previous result of subexponentiality of independent product (see Su and Chen, 2006) and also give some lemmas to be used in the proof of Theorem 2.1 in Section 3. Finally we give the proof of Theorem 2.1.

2. Main results

Throughout the paper we restrict ourselves to absolutely continuous $F \in \mathcal{S}$. For convenience, we write $X(F) \in \Gamma$, if the distribution $F$ of r.v. $X$ belongs to the class $\Gamma$. First we give our main results Theorem 2.1 as follows, and will give the proof in Section 4.

**Theorem 2.1.** Suppose that the assumptions $P_1$, $P_2$ and $P_3$ hold simultaneously. If

1. $F$ is absolutely continuous with a density function $f(\cdot)$ such that $xf(x)$ is eventually non-increasing, $F \in \mathcal{S}$;
2. for each fixed $a > 0$
   \[
   \lim_{x \to \infty} \frac{P(Y > x/a)}{P(XY > x)} = 0, \tag{2.1}
   \]
   then it holds for each $n = 1, 2, \ldots$ that
   \[
   \psi(x, n) \sim \sum_{k=1}^{n} P \left( X \prod_{j=1}^{k} Y_j > x \right). \tag{2.2}
   \]

**Remark 2.1.** It is pointed that the condition that $F$ is absolutely continuous with a density function $f(\cdot)$ such that $xf(x)$ is eventually non-increasing is not too restrictive. If $F$ is absolutely continuous with a finite expectation $\mu$, then
\[
\mu = \int_{0}^{\infty} xf(x) \, dx < \infty,
\]
thus we usually have
\[
\lim_{x \to \infty} xf(x) = 0.
\]
In fact, almost all useful subexponential distributions including the distributions with regularly varying tails, Lognormal, Pareto, Weibull, Loggamma, etc., satisfying the conditions above.

We will give some sufficient conditions for (2.1) below. We will see that the condition (2.1) is very rather relaxed and that essentially Theorem 2.1 generalize the results of Theorem A.

**Proposition 2.1.** Following the same notations of Theorem 2.1, if
\[
\lim_{x \to \infty} \frac{\xi(cx)}{F(x)} = 0 \tag{2.3}
\]
for some $0 < c < \infty$, then condition (2.1) holds for any $a > 0$. 


Proof. Only consider the case that $G$ is unbounded, i.e., $G(x) > 0$ for any $x > 0$. It is easy to see that for any $a > 0$,
\[
\limsup_{x \to \infty} \frac{G(x/a)}{P(XY > x)} \leq \limsup_{x \to \infty} \frac{G(x/a)}{\int_{a}^{\infty} \frac{G(x/y)}{F(x/y)} dy} \leq \frac{1}{G(ac)} \limsup_{x \to \infty} \frac{G(x/a)}{F(x/ac)} = 0,
\]
which implies that $\lim_{x \to \infty} \frac{G(x/a)}{P(XY > x)} = 0$ for any $a > 0$. □

By Proposition 2.1, we know the precise estimate of ruin probability (2.2) holds in the case that the tail of the insurance risk $X$ is heavier than that of the financial risk $Y$, particularly, for the case that $Y$ has a light tail.

In order to demonstrate that many popular distributions satisfying the condition (2.1), we introduce the definition of the class $\mathcal{R}/C_0^1$, which was first applied to the ruin theory by Tang and Tsitsashvili (2004). We say a distribution $F$ is said to have a rapid variation, denoted by $F \in \mathcal{R}/C_0^1$, if
\[
\lim_{x \to \infty} \frac{F(tx)}{F(x)} = 0 \quad \text{for any } t > 1.
\]
For the detailed properties of the class $\mathcal{R}/C_0^1$, we refer the reader to Bingham et al. (1987, Chapter 2.4), Embrechts et al. (1997, Chapter 3.3) or Tang and Tsitsashvili (2004).

By the definition of the class $\mathcal{R}/C_0^1$, it follows immediately:

**Proposition 2.2.** Suppose $G \in \mathcal{R}/C_0^1$, $F$ is unbounded, then the condition (2.1) holds for any $a > 0$.

**Proof.** For any fixed $a > 0$, by the definition of $\mathcal{R}/C_0^1$ we have
\[
\limsup_{x \to \infty} \frac{G(x/a)}{P(XY > x)} \leq \limsup_{x \to \infty} \frac{G(x/a)}{\int_{2a}^{\infty} G(x/y) dy} \leq \frac{1}{F(2a)} \limsup_{x \to \infty} \frac{G(x/a)}{G(x/2a)} = 0,
\]
we know the condition (2.1) holds for any $a > 0$. □

It is well-known that all power moments of $G$ are finite if $G \in \mathcal{R}/C_0^1$. And class $\mathcal{R}/C_0^1$ includes not only some heavy-tailed distributions such as the Lognormal distribution, Weibull distribution, etc. but also almost all light-tailed distributions. From Proposition 2.2, result (2.2) holds if only the financial risk $Y$ has a rapid varying tail, whether the tail of the insurance risk $X$ is heavier than that of the financial risk $Y$ or not.

### 3. Some lemmas

Subexponentiality of independent product plays an important role in time series, risk theory and stochastic process, which was pointed by Cline and Samorodnitsky (1994). The results about independent product are fewer, see Embrechts and Goldie (1980, 1982), Cline and Samorodnitsky (1994), Tang and Tsitsashvili (2003, 2004), Su and Chen (2006), etc. Su and Chen (2006) have proved the following proposition, which is crucial for proving Theorem 2.1.

**Proposition 3.3.** Let $X$ and $Y$ be independent non-negative r.v.’s with distribution $F$ and $G$, respectively. Suppose that $F$ is absolutely continuous with a density function $f(\cdot)$ and $xf(x)$ is eventually non-increasing. If $F \in \mathcal{F}$, and for each fixed $a > 0$,
\[
\lim_{x \to \infty} \frac{G(x/a)}{P(XY > x)} = 0, \quad (3.1)
\]
then $XY \in \mathcal{F}$.

In what follows we give the equivalent condition of the condition (3.1), which will be used many times throughout this paper.
**Proposition 3.4.** The relation (3.1) holds for each $a > 0$ if and only if the relation

$$\lim_{x \to \infty} \frac{\overline{G}(x/a(x))}{P(X > x)} = 0$$

holds for some positive function $a(\cdot)$ satisfying $a(x) \uparrow \infty$ and $x/a(x) \uparrow \infty$.

**Proof.** The sufficiency holds trivially, so we only need to prove the necessity. By (3.1), for any $n \geq 1$, choosing $a_n$ satisfying $2^n < a_n < 2^{n+1}$, then there exists $x_n$ such that when $x > x_n$,

$$\frac{\overline{G}(x/a_n)}{P(X > x)} \leq 2^{-n}.$$ 

Choosing $\bar{x}_1 = \max\{1, x_1\}, \bar{x}_n = \max\{2^{n+1} \bar{x}_{n-1}, x_n\}, n \geq 2$, define

$$a(x) = \begin{cases} 1, & 0 < x < \bar{x}_1, \\ a_n, & \bar{x}_n \leq x < \bar{x}_{n+1}, & n \geq 1. \end{cases}$$

By the above definition we know $a(x) \to \infty$. If $0 < x < \bar{x}_1$, $x/a(x) = x$. If $\bar{x}_n \leq x < \bar{x}_{n+1}$,

$$x/a(x) = x/a_n \geq 2^{-(n+1)} \bar{x}_n \geq \bar{x}_{n-1} \to \infty \quad \text{as} \quad n \to \infty$$

and

$$\frac{\overline{G}(x/a(x))}{P(X > x)} = \frac{\overline{G}(x/a_n)}{P(X > x)} \leq 2^{-n}, \quad n \geq 1.$$ 

So

$$\frac{\overline{G}(x/a(x))}{P(X > x)} \to 0 \quad \text{as} \quad x \to \infty. \quad \square$$

Tang and Tsitsashvili (2003, Lemma 3.2) proved the following result:

**Lemma 3.1.** Let $F_1$ and $F_2$ be two d.f.'s concentrated on $(-\infty, \infty)$. If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{S}$, and $\overline{F}_2(x) = O(\overline{F}_1(x))$, then $F_1 \ast F_2 \in \mathcal{S}$ and

$$\overline{F}_1 \ast \overline{F}_2(x) \sim \overline{F}_1(x) + \overline{F}_2(x).$$

**Lemma 3.2.** Let $F_1$ be a d.f. supported on $(-\infty, \infty)$ and $F_2$ be a finite non-negative measure. If $F_1 \in \mathcal{L}$ and $F_1 \sim F_2$, then $F_1 \ast F_1 \sim F_2 \ast F_2$.

**Proof.** See Corollary 2.5 in Cline (1987). \square

**Lemma 3.3.** Let $F_1$ and $F_2$ be two d.f.'s concentrated on $(-\infty, \infty)$. Suppose $F_1 \in \mathcal{S}, F_2 \in \mathcal{S}$, and $F_1 \ast F_2 \sim F_1 + F_2$, then $F_1 \ast F_2 \in \mathcal{S}$.

**Proof.** First it is easy to see $F_1 \ast F_2 \in \mathcal{L}$, then by Lemma 3.2, we have

$$\frac{1}{(F_1 \ast F_2)^2} \sim \frac{F_1 \ast F_1}{2} + \frac{2F_1 \ast F_2 + F_2 \ast F_2}{2}.$$

Then we have $F_1 \ast F_2 \in \mathcal{S}$ by the definition of the class $\mathcal{S}$. \square

**Lemma 3.4.** Suppose that $F$ is absolutely continuous with a density function $f(\cdot)$ such that $xf(x)$ is eventually non-increasing, $(Y_1, Y_n, n \geq 1)$ is i.i.d. r.v.'s series with the generic distribution $G$ and independent of $X$. If $F \in \mathcal{S}$, and (3.1) holds for each fixed $a > 0$, then for any $n \geq 1$ we have $X \prod_{j=1}^{n} Y_j \in \mathcal{S}$. 

\[\square\]
Lemma 3.5. Under the same conditions of Theorem 2.1, then we have that $V_n Y \in \mathcal{S}$ for all $n \geq 1$, where $V_n, n \geq 1$ is defined by (3.5).
**Proof.** By Lemma 3.4, we know \( XY_1 Y_2 \in \mathcal{S} \), that is \( V_1 Y_2 \in \mathcal{S} \). Now we assume by induction that \( V_{k-1} Y_k \in \mathcal{S} \), \( k \geq 2 \), then we will prove \( V_k Y_{k+1} \in \mathcal{S} \). By (3.5) we know for any \( x > 0 \),

\[
P(V_k Y_{k+1} > x) = P((X_k + V_{k-1}) Y_k Y_{k+1} > x) = P\left( \sum_{j=1}^{k} X_j \prod_{i=j}^{k+1} Y_i > x \right)
\]

\[
= P\left( \sum_{j=2}^{k} X_j \prod_{i=j}^{k+1} Y_i + X \prod_{i=1}^{k+1} Y_i > x \right) = P\left( V_{k-1} Y_k + X \prod_{i=1}^{k+1} Y_i > x \right),
\]

the last equation follows by \( \sum_{j=2}^{k} X_j \prod_{i=j}^{k+1} Y_i \equiv V_{k-1} Y_k \).

1°. \( P(0 < Y \leq 1) = 1 \), for any \( l \leq k, k \geq 1 \),

\[
P\left( X \prod_{i=1}^{l} Y_i > x \right) = O\left( P\left( X \prod_{i=1}^{l} Y_i > x \right) \right).
\]

It follows that \( P(X \prod_{i=1}^{k+1} Y_i > x) = O(P(V_{k-1} Y_k > x)), k \geq 2 \).

2°. \( P(Y > 1) > 0 \), thus for any \( l \leq k, k \geq 1 \),

\[
P\left( X \prod_{i=1}^{l} Y_i > x \right) = \int_{0}^{\infty} P\left( X \prod_{i=1}^{l} Y_i > x/t \right) dP\left( \prod_{i=l+1}^{k} Y_i \leq t \right)
\]

\[
\geq \int_{1}^{\infty} P\left( X \prod_{i=1}^{l} Y_i > x/t \right) dP\left( \prod_{i=l+1}^{k} Y_i \leq t \right)
\]

\[
\geq P\left( X \prod_{i=1}^{l} Y_i > x \right) P\left( \prod_{i=l+1}^{k} Y_i > 1 \right).
\]

So it follows that

\[
P\left( X \prod_{i=1}^{l} Y_i > x \right) = O\left( P\left( X \prod_{i=1}^{l} Y_i > x \right) \right),
\]

that is \( P(V_{k-1} Y_k > x) = O(P(X \prod_{i=1}^{k+1} Y_i > x)), k \geq 2 \).

By Lemma 3.4 we know \( X \prod_{i=1}^{k+1} Y_i \in \mathcal{S} \), thus applying Lemma 3.1 we know \( V_k Y_{k+1} \in \mathcal{S} \). This completes the proof. \( \square \)

4. **The proof of Theorem 2.1**

The proof of Theorem 2.1 can be formulated into two parts according to \( P(Y > 1) = 0 \) and \( P(Y > 1) > 0 \), respectively.

1° \( P(0 < Y \leq 1) = 1 \). By (3.6), it is clear that

\[
\psi(x, 1) = P(V_1 > x) = P(X_1 Y_1 > x), \quad x > 0.
\]

(4.1)

From Proposition 3.3 we know \( V_1 \in \mathcal{S} \) and (2.2) holds for \( n = 1 \). It is trivial that \( P(V_1 > x) = O(\mathcal{F}(x)) \) for \( x > 0 \).

Now we assume by induction that (2.2) holds for \( n = m, V_m \in \mathcal{S}, m \geq 1 \) and

\[
P(V_m > x) = O(\mathcal{F}(x)), \quad x > 0.
\]

Then we need to show the above three facts hold with \( n = m + 1 \). By the assumption \( V_m \in \mathcal{S} \) and \( F \in \mathcal{S} \), applying Lemma 3.1 we know for \( x > 0 \),

\[
P(V_{m+1} > x) = P((X_{m+1} + V_m) Y_{m+1} > x) \leq P(X_{m+1} + V_m > x)
\]

\[
\sim P(X_{m+1} > x) + P(V_m > x) = O(\mathcal{F}(x)).
\]
Now we prove (2.2) holds for \( n = m + 1 \), for \( x > 0 \),
\[
\psi(x, m + 1) = \int_0^1 P(X_{m+1} + V_m > x/t) G(dt)
\sim \int_0^1 (P(X_{m+1} > x/t) + P(V_m > x/t)) G(dt)
\approx P(X_{m+1} > t) + \int_0^1 \sum_{j=1}^m P \left( X \prod_{j=1}^k Y_j > x/t \right) G(dt)
= \sum_{k=1}^{m+1} P \left( X \prod_{j=1}^k Y_j > x \right).
\]

From (4.2), we have for \( x > 0 \) that
\[
P(V_{m+1} > x) = P((X_{m+1} + V_m) Y_{m+1} > x) \sim P(X_{m+1} Y_{m+1} > x) + P(V_{m} Y_{m+1} > x).
\]

This with Lemmas 3.4 and 3.5 implies \( X_{m+1} Y_{m+1} \in \mathcal{I} \) and \( V_m Y_{m+1} \in \mathcal{I} \), thus applying Lemma 3.3 we obtain \( V_{m+1} \in \mathcal{I} \).

2° \( P(Y > 1) > 0 \). In this case it is obvious that \( \mathcal{F}(x) = O(P(V_1 > x)) \) for \( x > 0 \). Similar to 1°, we have \( V_1 \in \mathcal{I} \) and (2.2) holds for \( n = 1 \).

Now assume by induction that (2.2) holds for \( n = m, V_m \in \mathcal{I}, \ m \geq 2 \) and \( \mathcal{F}(x) = O(P(V_m > x)) \) for \( x > 0 \). By the assumption of the induction, we have for \( x > 0 \),
\[
P(X_{m+1} + V_m > x) \sim P(X_{m+1} > x) + P(V_m > x).
\]

By (4.3), for \( x > 0 \) we have
\[
P(V_{m+1} > x) = P((X_{m+1} + V_m) Y_{m+1} > x) = \int_0^\infty P \left( X_{m+1} + V_m > \frac{x}{y} \right) G(dy)
\geq \int_1^\infty P \left( X_{m+1} + V_m > \frac{x}{y} \right) G(dy)
\geq P(X_{m+1} + V_m > x) \mathcal{G}(1)
\sim (P(X_{m+1} > x) + P(V_m > x)) \mathcal{G}(1)
\geq P(V_m > x) \mathcal{G}(1)
\geq \cdots \geq P(V_1 > x) \mathcal{G}(1)^m \geq \mathcal{F}(x) \mathcal{G}(1)^m,
\]
that is \( \mathcal{F}(x) = O(P(V_{m+1} > x)) \).

Finally, we will prove (2.2) holds for \( n = m + 1 \) and \( V_{m+1} \in \mathcal{I} \). By Lemma 3.1,
\[
\psi(x, m + 1) = P(V_{m+1} > x) = P((X_{m+1} + V_m) Y_{m+1} > x)
= \left( \int_0^{x/a(x)} + \int_{x/a(x)}^\infty \right) P \left( X_{m+1} + V_m > \frac{x}{y} \right) G(dy)
\approx J_1 + J_2.
\]
From (4.3) and (3.2),
\[
J_1 = \int_0^{x/a(x)} P(X_{m+1} + V_m > x/y) G(dy)
\sim \int_0^{x/a(x)} (P(X_{m+1} > x/y) + P(V_m > x/y)) G(dy)
\]

(4.4)
\[
\sim \left( \int_0^\infty - \int_x^{x/a(x)} \right) \left( \sum_{k=1}^m P(X_{m+1} > x/y) + \sum_{k=1}^m P \left( X \prod_{i=1}^k Y_i > x/y \right) \right) G(dy) \\
= P(X_{m+1}Y_{m+1} > x) + \sum_{k=1}^m P \left( X \prod_{i=1}^{k+1} Y_i > x \right) + O(1) \bar{G} \left( \frac{x}{a(x)} \right)
\]

On the other hand,

\[
J_2 = O(1) \bar{G} \left( \frac{x}{a(x)} \right) = o(1) P(XY > x).
\]

Following (4.4), for \(x > 0\),

\[
P(V_{m+1} > x) = P((X_{m+1} + V_m) Y_{m+1} > x)
\]

\[
= P(X_{m+1} Y_{m+1} > x) + P(V_m Y_{m+1} > x) + O(1) \bar{G} \left( \frac{x}{a(x)} \right)
\]

\[
\sim P(X_{m+1} Y_{m+1} > x) + P(V_m Y_{m+1} > x).
\]

By Lemmas 3.4 and 3.5 we know \(X_{m+1} \in \mathcal{F}\) and \(V_m Y_{m+1} \in \mathcal{F}\), thus by applying Lemma 3.3 it follows that \(V_{m+1} \in \mathcal{F}\).

Combining 1\(^{\circ}\) and 2\(^{\circ}\), the proof is completed. □

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