A general method to the strong law of large numbers and its applications

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Abstract

A general method to prove the strong law of large numbers is given by using the maximal tail probability. As a result the convergence rate of $S_n/n$ for both positively associated sequences and negatively associated sequences is $n^{-1/2}(\log n)^{\delta/2}(\log \log n)^{\delta/2}$ for any $\delta > 1$. This result closes to the optimal achievable convergence rate under independent random variables, and improves the rates $n^{-1/3}(\log n)^{2/3}$ and $n^{-1/3}(\log n)^{5/3}$ obtained by Ioannides and Roussas [1999. Exponential inequality for associated random variables. Statist. Probab. Lett. 42, 423–431] and Oliveira [2005. An exponential inequality for associated variables. Statist. Probab. Lett. 73, 189–197], respectively. In this sense the proposed general method may be more effective than its peers provided by Fazekas and Klesov [2001. A general approach to the strong law of large numbers. Theory Probab. Appl. 45(3), 436–449] and Ioannides and Roussas [1999. Exponential inequality for associated random variables. Statist. Probab. Lett. 42, 423–431].

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1. Introduction

Throughout the paper we suppose that \(\{X_j : j \geq 1\}\) is a sequence of random variables defined on the probability space \((\Omega, \mathcal{F}, P)\) and \(S_n = \sum_{j=1}^{n} X_j\). Fazekas and Klesov (2001) gave the following Theorem 1.1 on the strong law of large numbers by using the maximal inequality of Háyek-Rényi type.

\textbf{Theorem 1.1.} Let \(b_1, b_2, \ldots\) be a nondecreasing unbounded sequence of positive numbers. Let \(x_1, x_2, \ldots\) be nonnegative numbers. Let \(r\) be a fixed positive number. Assume that for each \(n \geq 1\)

\[E\left(\max_{1 \leq k \leq n} |S_k|\right)^r \leq \sum_{j=1}^{n} x_j. \tag{1.1}\]
If
\[ \sum_{j=1}^{\infty} a_j/b_j^p < \infty, \]  
then
\[ \lim_{n \to \infty} S_n/b_n = 0 \quad \text{a.s.} \]  

This theorem does not need any restrictions on the dependence structure of random variables and \( b_n \) is not assumed to satisfy any regularity conditions. As a result of this theorem, Fazekas and Klesov (2001) explored some strong laws of large numbers for martingales and \( \rho \)-mixing sequences, then Kuczmaszewska (2005) proved a strong law of large numbers for negatively associated sequences and extended the result to \( \rho \)-mixing sequences.

In most of applications of the strong law of large numbers, however, the sequence \( b_n \) in a partial sum usually satisfies or is assumed to satisfy some regularity conditions. In this paper, we will prove that a better rate of strong convergence than Theorem 1.1 can be obtained if one takes advantage of some regularity condition for \( b_n \).

Basically, we assume that \( b_n \) satisfies the regularity condition (2.1) in Section 2 and use the following (1.4) to replace (1.1) and (1.2) in Theorem 1.1:
\[ \sum_{n=1}^{\infty} n^{-1} P \left( \max_{1 \leq k \leq n} |S_k| > \varepsilon b_n \right) < \infty. \]

Under these changes which add nothing on the restriction of the dependence structure of random variables, the convergence rate of the strong laws of large numbers can be improved. Moreover, an effective method to obtain strong law of large numbers is provided in Sections 3 and 4. In fact, as an application of the effective method, we discuss strong laws of large numbers for both positively associated sequences and negatively associated sequences in Sections 3 and 4.

If the covariances of positively associated variables decay geometrically and random variables are bounded, Ioannides and Roussas (1999) used their exponential inequality to obtain the rate \( n^{-1/3}(\log n)^{5/3} \) for the strong law of large numbers \( S_n/n \), while Oliveira (2005) only got \( n^{-1/3}(\log n)^{2/3} \). In Section 3, we obtain the faster rates for positively associated random variables without the assumption that the random variables are bounded and their covariances decay geometrically. Our results imply that the convergence rate of \( S_n/n \) is \( n^{-1/2}(\log n)^{1/2}(\log \log n)^{3/2} \) for any \( \delta > 1 \) provided that \( \sup_{j \geq 1} E|X_j|^3 < \infty \) and the covariances of positively associated variables decrease polynomially, i.e. \( \text{Cov}(X_i, X_j) \leq C|j-i|^{-\delta} \text{log}(|j-i|)^{\delta} \) where \( \lambda > 3 \), the symbol \( C \) or \( c \) henceforth denotes any positive constant independent of \( n \) but whose values may vary over cases. The result closes to the optimal achievable convergence rate for independent random variables under an iterated logarithm.

The paper is organized as follows. Section 2 describes our general result on strong law of large numbers. Sections 3 and 4 consider the strong laws of large numbers for both positively associated sequences and negatively associated sequences.

2. A general method to the strong law of large numbers

Here we first give a general theorem to the strong law of large numbers.

**Theorem 2.1.** Let \( b_1, b_2, \ldots \) be a nondecreasing sequence of positive numbers with
\[ 1 \leq b_{2n}/b_n \leq c < \infty \]
for some \( c > 1 \). Assume that for any given \( \varepsilon > 0 \),
\[ \sum_{n=1}^{\infty} n^{-1} P \left( \max_{1 \leq k \leq n} |S_k| > b_n \varepsilon \right) \leq C < \infty, \]

\[ \sum_{j=1}^{\infty} a_j/b_j^p < \infty, \] (1.2)
then
\[
\lim_{n \to \infty} \frac{\max_{1 \leq j \leq n} |S_j|}{b_n} = 0 \quad a.s.
\] (2.3)

Before proving the theorem, we make some comments about it.

**Remark 2.1.** As sequence \( \max_{1 \leq j \leq n} |S_j| \) is always nondecreasing with respect to \( n \), it follows that \( \{b_n\} \) is a nondecreasing sequence from (2.3). Furthermore, \( \max_{1 \leq j \leq n} |S_j| \) in Theorem 2.1 may be replaced by an arbitrary nondecreasing and nonnegative sequence of random variables, we adapt it simply due to comparison with Theorem 1.1.

**Remark 2.2.** One usually takes \( b_n = n^2 \ell(n) \) where \( \ell(n) = n^\alpha \) is a slowly varying sequence and nondecreasing, such as \( \ell(n) = (\log n)^\beta \log \log n \delta \) for some \( \beta, \delta > 0 \). In this case, \( b_n \) is nondecreasing and satisfies \( 1 \leq b_{2n}/b_n \leq 2^x + 1 \) by noting that \( \ell(2n)/\ell(n) \to 1 \), i.e. (2.1) holds. But if \( b_n \) tends to infinity faster than \( n^\alpha \ell(n) \), then (2.1) may not hold. For example, if \( b_n \) is geometrically increasing, such as \( b_n = \rho^n \) for some \( \rho > 1 \), then \( b_{2n}/b_n \) is increasing, but unbounded. In the case, however, (2.3) holds under an alternative moment condition; see the detail from the Theorem 2.2 below.

**Theorem 2.2.** Let \( b_1, b_2, \ldots \) be a nondecreasing sequence of positive numbers such that \( b_{2n}/b_n \) is increasing and unbounded. If \( \sup_{j \geq 1} E|X_j|^r < \infty \) for some \( 0 < r < 1 \), then (2.3) holds.

**Proof.** From the fact that \( b_{2n}/b_n \) is increasing and unbounded, we have that for any given \( M > 0 \), \( b_{2n}/b_n \geq M \) for sufficiently large \( n \). Without loss of generality, assume that \( b_{2^k}/b_{2^{k-1}} \geq M \) for all \( k > 1 \), then \( b_{2^k} \geq Mb_{2^{k-1}} \geq \cdots \geq M^k b_1 \). Taking \( M = 5^{1/r} \), we have

\[
\sum_{n=1}^\infty P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon b_n \right) \leq C \sum_{n=1}^\infty b_n^{-r} E \max_{1 \leq j \leq n} |S_j|^r \\
\leq C \sum_{n=1}^\infty b_n^{-r} \sum_{j=1}^n E|X_j|^r \leq C \sum_{n=1}^\infty nb_n^{-r} \\
= C \sum_{k=1}^\infty \sum_{2^{k-1} \leq n < 2^k} nb_n^{-r} \leq C \sum_{k=1}^\infty \sum_{2^{k-1} \leq n < 2^k} 2^k b_n^{-r} \\
= C \sum_{k=1}^\infty 2^{(k-1)} b_{2^{k-1}}^{-r} \leq C \sum_{k=1}^\infty 2^{(k-1)} M^{-r(k-1)} \\
= C \sum_{k=1}^\infty (4/M^r)^{k-1} < \infty.
\]

So (2.3) holds. \( \Box \)

**Remark 2.3.** The conditions (1.1) and (1.2) of Theorem 1.1 imply that (2.2) of Theorem 2.1 usually holds in most of the cases. In fact, by Markov’s inequality and (1.1), we know that

\[
\sum_{n=1}^\infty n^{-1} P \left( \max_{1 \leq k \leq n} |S_k| > b_n \right) \leq C \sum_{n=1}^\infty n \sum_{j=1}^n \alpha_j n^{-1} b_n^{-r}.
\]

(1) If \( \{\alpha_j : j \geq 1\} \) is nondecreasing, we have

\[
\sum_{n=1}^\infty n \sum_{j=1}^n \alpha_j n^{-1} b_n^{-r} \leq \sum_{n=1}^\infty \alpha_n b_n^{-r} < \infty.
\]
(2) If \( b_n = n^\alpha \ell(n) \) for some \( \alpha > 0 \) where \( \ell(n) \) is a slowly varying sequence and nondecreasing, then

\[
\sum_{n=1}^{\infty} n^{-1} b_n^{-\gamma} = \sum_{n=1}^{\infty} n^{-1-\gamma \ell(n)} \leq j^{-\gamma \ell(j)/2} (\ell(j))^{-\gamma} \sum_{n=1}^{\infty} n^{-1-\gamma/2} \leq C j^{-\gamma/2} (\ell(j))^{-\gamma} j^{-\gamma/2} = C b_j^{-\gamma},
\]

so

\[
\sum_{n=1}^{\infty} \sum_{j=1}^{n} a_j n^{-1} b_n^{-\gamma} = \sum_{j=1}^{\infty} a_j \sum_{n=1}^{\infty} n^{-1} b_n^{-\gamma} \leq C \sum_{j=1}^{\infty} a_j / b_j < \infty.
\]

Now, we consider to prove Theorem 2.1.

**Proof of Theorem 2.1.** By \( b_n \leq b_{2n} \leq c b_n \), we have

\[
\sum_{k=1}^{\infty} P \left( \max_{1 \leq j \leq 2^k} |S_j| > \varepsilon b_{2^k} \right)
\]

\[
= \sum_{k=1}^{\infty} \sum_{n=1}^{n} (2^k)^{1-1} P \left( \max_{1 \leq j \leq 2^k} |S_j| > \varepsilon b_{2^k} \right)
\]

\[
\leq \sum_{k=1}^{\infty} \sum_{n=1}^{n} (n/2)^{-1} P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon b_{n/2} \right)
\]

\[
+ \sum_{k=1}^{\infty} \sum_{n=1}^{n} (n/2)^{-1} P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon b_{(n+1)/2} \right)
\]

\[
\leq 2 \sum_{k=1}^{\infty} \sum_{n=1}^{n} n^{-1} P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon^{-1} b_n \right)
\]

\[
+ 2 \sum_{k=1}^{\infty} \sum_{n=1}^{n} n^{-1} P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon^{-1} b_{n+1} \right)
\]

\[
\leq 2 \sum_{k=1}^{\infty} \sum_{n=1}^{n} n^{-1} P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon^{-1} b_{n} \right)
\]

\[
= 2 \sum_{n=1}^{\infty} n^{-1} P \left( \max_{1 \leq j \leq n} |S_j| > \varepsilon^{-1} b_{n} \right) < \infty.
\]

By the Borel–Cantelli lemma, this implies that

\[
\max_{1 \leq j \leq 2^k} |S_j| / b_{2^k} \to 0 \quad \text{a.s.} \quad (k \to \infty).
\]

Furthermore,

\[
\max_{2^{k-1} < n \leq 2^k} \max_{1 \leq j \leq n} |S_j| / b_n \leq \max_{2^{k-1} < n \leq 2^k} \max_{1 \leq j \leq 2^k} |S_j| / b_{2^k-1} \leq \varepsilon^{-1} \max_{2^{k-1} < n \leq 2^k} \max_{1 \leq j \leq 2^k} |S_j| / b_{2^k-1}
\]

\[
= \varepsilon^{-1} \max_{1 \leq j \leq 2^k} |S_j| / b_{2^k} \to 0 \quad \text{a.s.} \quad (k \to \infty).
\]

So from the sub-sequence method, we obtain that

\[
\max_{1 \leq j \leq n} |S_j| / b_n \to 0 \quad \text{a.s.} \quad (n \to \infty).
\]

Theorem 2.1 is proved. □
3. Strong law of large numbers for PA sequence

In this section, we will apply Theorem 2.1 to prove some strong laws of large numbers for positively associated random variables.

**Definition 3.1.** The random variables $X_1, X_2, \ldots$ are said to be positively associated if for every $n$ and $f, g : \mathbb{R}^n \to \mathbb{R}$ coordinatewise increasing,

$$\text{Cov}(f(X_1, X_2, \ldots, X_n), g(X_1, X_2, \ldots, X_n)) \geq 0,$$

whenever this covariance exists.

**Theorem 3.1.** Let $\{X_j : j \geq 1\}$ be a sequence of positively associated random variables with

$$X_1 \sim \chi^2_n,$$

where $u(n) = \sup_{j \geq n} \sum_{j_{k-1} \geq n} \text{Cov}(X_i, X_j)$. Let $\varphi : \mathbb{R} \to \mathbb{R}^+$ be a function which is even and nondecreasing on $[0, \infty)$ with $\lim_{x \to \infty} \varphi(x) = \infty$, and such that

(a) $\varphi(x) / x \downarrow$, or
(b) $\varphi(x) / x \uparrow$ and $\varphi(x) / x^2 \downarrow$, $x \to \infty$ and $EX_j = 0$.

Assume further that $b_1, b_2, \ldots$ is a nondecreasing sequence of positive numbers satisfying

$$1 \leq b_{2n}/b_n \leq c < \infty \quad \text{for all } n \geq 1,$$

$$\sum_{n=1}^{\infty} b_n^{-2} < \infty,$$

$$\sum_{n=1}^{\infty} \frac{\text{E} \varphi(X_n)}{\varphi(b_n)} < \infty,$$

$$\sum_{n=1}^{\infty} b_n^{-2} \max_{1 \leq j \leq n} b_j^2 \text{P}(|X_j| > b_j) < \infty,$$

$$\sum_{n=1}^{\infty} b_n^{-2} \max_{1 \leq j \leq n} \frac{b_j^2 \text{E} \varphi(X_j) \text{I}(|X_j| \leq b_j)}{\varphi(b_j)} < \infty,$$

then

$$S_n/b_n \to 0 \quad \text{a.s.}$$

**Remark 3.1.** Let $u(n) = 2\sum_{j=n+1}^{\infty} \text{Cov}(X_i, X_j)$ for stationary random variables. At that case, if $u(n) \leq C (\log n)^{-2} (\log \log n)^{-3}$, then (3.1) is satisfied. In general, the decrease rate of $u(n)$ is often assumed as $n^{-\rho}$ for some $\rho > 0$ (e.g., Birkel, 1988; Shao and Yu, 1996). Furthermore, if the covariances decrease polynomially, i.e. $\text{Cov}(X_i, X_j) \leq C |j-i|^{-1} (\log |j-i|)^{l}$ where $\lambda > 3$, then (3.1) holds. So (3.1) is a very weak condition. The mild conditions (3.3)–(3.6) could be replaced by other conditions. In fact, we have the following Corollary 3.1.

**Corollary 3.1.** If conditions (3.3)–(3.6) are replaced by

$$\sup_{j \geq 1} \text{E} \varphi(X_j) < \infty$$

and

$$\sum_{n=1}^{\infty} \varphi^{-1}(b_n) < \infty,$$

then (3.7) still holds.
Proof. Clearly, $x^2/\varphi(x)$ is nondecreasing for both cases (a) and (b) due to $\varphi(x)/x \searrow$ in the case (a) and $\varphi(x)/x^2 \searrow$ in the case (b). Note that $b_n$ is nondecreasing; we have
\begin{equation}
0 < \frac{b_n^2}{\varphi(b_n)} \leq \frac{b_n^2}{\varphi(b_1)} \quad (3.10)
\end{equation}
and
\begin{equation}
\max_{1 \leq j \leq n} \frac{b_n^2}{\varphi(b_j)} \leq \frac{b_n^2}{\varphi(b_n)}, \quad (3.11)
\end{equation}
Hence, we can derive (3.3)–(3.6) from (3.8)–(3.11) and Minkowski’s inequality for (3.5). \qed

Take $\varphi(x) = |x|^p$ and $b_n = (n \log n (\log \log n)^3)^{1/p}$ in Corollary 3.1, we immediately obtain the following result.

Corollary 3.2. Assume that $\{X_j : j \geq 1\}$ is a sequence of positively associated random variables with zero means and $\sup_{j \geq 1} E|X_j|^p < \infty$ for some $1 \leq p \leq 2$, and satisfies (3.1). Then for any $\delta > 1,$
\begin{equation}
S_n/(n \log n (\log \log n)^3)^{1/p} \to 0 \quad \text{a.s.} \quad (3.12)
\end{equation}

Remark 3.2. For the case $p = 2$, Corollary 3.2 implies that the convergence rate of $S_n/n$ is $n^{-1/2}(\log n)^{5/2}$. The result closes to the optimal rate obtained under the iterated logarithm for independent random variables.

Under the conditions that random variables are bounded and the covariances decay geometrically, Ioannides and Roussas (1999) obtained the rate $n^{-1/3}(\log n)^{2/3}$ for $S_n/n$ and Oliveira (2005) got $n^{-1/3}(\log n)^{2/3}$. Without these conditions Corollary 3.2 provides better rates. In addition, condition (3.1) is very flexible for positively associated random variables. In fact, it only needs to decay polynomially, does not need to decrease geometrically, see the details from Remark 3.1.

We now prove Theorem 3.1. To get the upper bound of the tail probability of maximal sums $P(\max_{1 \leq k \leq n} |S_k| > b_n)$ in Theorem 2.1, some moment inequalities are needed.

Lemma 3.1 (Newman and Wright, 1981, Theorem 2). Let $X_1, X_2, \ldots$ be positively associated random variables with zero means and $EX^2_j < \infty.$ Then
\begin{equation}
E \max_{1 \leq j \leq n} S_j^2 \leq ES_n^2. \quad (3.13)
\end{equation}

Lemma 3.2. Let $X_1, X_2, \ldots$ be positively associated random variables with zero means, $EX^2_j < \infty$ and satisfying (3.1). Then there exists a positive constant $C$, which does not depend on $n$, such that
\begin{equation}
E \max_{1 \leq j \leq n} S_j^2 \leq Cn \left( \max_{1 \leq j \leq n} EX^2_j + 1 \right). \quad (3.14)
\end{equation}

Proof. For each given $n \geq 1$, let
\begin{equation}
Y_j = \begin{cases} X_j, & 1 \leq j \leq n, \\ 0, & j > n, 
\end{cases}
\end{equation}
and $Y_1, Y_2, \ldots, Y_n$ have the same dependent structures as $X_1, X_2, \ldots, X_n.$ Denote $\|Y\|_2 = (EY^2)^{1/2}, S_k(n) = \sum_{i=k+1}^{k+n} Y_i$ and $\sigma_n = \sup_{k \geq 0} \|S_k(n)\|_2.$ Obviously, $\sigma^2_n = \max_{1 \leq j \leq n} EX^2_j$ and
\begin{equation}
S_k(2m) = S_k(m) + S_{k+m+[m^{1/3}])(m) + S_{k+m}(m^{1/3}) - S_{k+2m}(m^{1/3}).
\end{equation}

Hence, by Minkowski’s inequality, we have
\begin{align*}
\|S_k(2m)\|_2 & \leq \|S_k(m) + S_{k+m+[m^{1/3}])(m)\|_2 + \|S_{k+m}(m^{1/3})\|_2 + \|S_{k+2m}(m^{1/3})\|_2 \\
& \leq \|S_k(m) + S_{k+m+[m^{1/3}])(m)\|_2 + 2m^{1/3} \sigma_1.
\end{align*}
Also, since $EY_i = 0$ and $\text{Cov}(Y_i, Y_j) \geq 0$,

\[
E(S_k(m) + S_{k+m+[m/3]}(m))^2
= ES_k^2(m) + ES_k^2_{k+m+[m/3]}(m) + 2ES_k(m)S_{k+m+[m/3]}(m)
\leq 2\sigma_m^2 + 2 \sum_{j=k+1}^{k+m} \sum_{j=k+m+[m/3]+1}^{\infty} \text{Cov}(Y_i, Y_j)
\leq 2\sigma_m^2 + 2 \sum_{j=k+1}^{k+m} u([m/3])
\leq 2\sigma_m^2 + 2mu([m/3]).
\]

Therefore,

\[
\sigma_{2m} \leq 2^{1/2}\sigma_m + [2mu([m/3])]^{1/2} + 2m^{1/3}\sigma_1.
\]

Taking $m = 2^{r-1}$ in the equation above,

\[
\sigma_2 \leq 2^{1/2}\sigma_{2^{r-1}} + 2 \cdot 2^{(r-1)/3} \sigma_1 + 2^{r/2}u^{1/2}([2^{(r-1)/3}]).
\]

Using this recursive inequality, we get

\[
\sigma_2 \leq 2^{r/2}\sigma_1 + 2\sigma_1 \sum_{i=0}^{r-1} 2^{(r-1-i)/2}i^{1/3} + 2^{r/2}u^{1/2}([2^{i/3}])
\leq 2^{r/2} + 2\sum_{i=0}^{r-1} 2^{(r-1-i)/2}i^{1/3} + 2^{r/2}u^{1/2}([2^{i/3}])
\leq 2^{r/2} + 2^{(r+1)/2} \sum_{i=0}^{\infty} 2^{-i/6} + 3 \cdot 2^{r/2} \sum_{i=0}^{\infty} u^{1/2}(2^i)
\leq C2^{r/2}(\sigma_1 + 1)
\]

by the fact that $u(n)$ is decreasing with respect to $n$. Thus, we have

\[
\sigma_2 \leq C2^r(\sigma_1^2 + 1).
\]

Note that for each given $n \geq 1$, there exists a positive integer $r \geq 0$ such that $2^r \leq n < 2^{r+1}$, we obtain further

\[
ES_n^2 \leq ES_0^2(2^{r+1}) \leq \sigma_{2^{r+1}}^2 \leq C2^{r+1}(\sigma_1^2 + 1) \leq 2Cu(\sigma_1^2 + 1).
\]

Therefore, (3.14) follows from the equation above and Lemma 3.1. □

**Proof of Theorem 3.1.** Let $Z_j = X_jI(|X_j| \leq b_j) - b_jI(X_j < -b_j) + b_jI(X_j > b_j)$. We first show that

\[
b_n^{-1} \left| E \sum_{j=1}^{n} Z_j \right| \rightarrow 0. \quad (3.15)
\]

If $\varphi(x)$ satisfies condition (a), i.e. $\varphi(x)/x$ \&, then $\varphi(b_j)/b_j \leq \varphi(|X_j|)/|X_j|$ whenever $|X_j| \leq b_j$, so $|X_j|/b_j \leq \varphi(X_j)/\varphi(b_j)$. Hence

\[
|EZ_j| \leq b_jE(|X_j|)/b_jI(|X_j| \leq b_j) + b_jP(|X_j| > b_j)
\leq b_jE(\varphi(|X_j|)/\varphi(b_j))I(|X_j| \leq b_j) + b_jE\varphi(X_j)/\varphi(b_j)
\leq 2b_jE\varphi(X_j)/\varphi(b_j).
\]
If \( \varphi(x) \) satisfies condition (b), i.e. \( \varphi(x)/x \not\nearrow \), then \( \varphi(|X|)/|X| \geq \varphi(b)/b \) whenever \( |X| > b \), so \( |X|/b \leq \varphi(X)/\varphi(b) \). And note that \( EX_j = 0 \); we get

\[
|EZ_j| = |EX_jI(|X| \leq b)| + b_jP(|X| > b_j)
\]

\[
= |EX_jI(|X| > b_j)| + b_jP(|X| > b_j)
\]

\[
\leq b_jE(|X|)/b_jI(|X| > b_j) + b_jP(|X| > b_j)
\]

\[
\leq b_jE(\varphi(|X|))/\varphi(b_j)I(|X| > b_j) + b_jE\varphi(X)/\varphi(b_j)
\]

\[
\leq 2b_jE\varphi(X)/\varphi(b_j).
\]

So we have that

\[
|EZ_j| \leq 2b_jE\varphi(X)/\varphi(b_j) \tag{3.16}
\]

for both cases. From (3.16) and (3.4), we get \( \sum_{j=1}^{\infty}|EZ_j|/b_j < \infty \). Therefore \( b_n^{-1}\sum_{j=1}^{n}|EZ_j| \to 0 \) by the Kronecker lemma, so (3.15) is true. On the other hand, from (3.4)

\[
\sum_{j=1}^{\infty}P(X_j \neq Z_j) = \sum_{j=1}^{\infty}P(|X_j| > b_j) \leq \sum_{j=1}^{\infty}\frac{E\varphi(X_j)}{\varphi(b_j)} < \infty. \tag{3.17}
\]

By (3.15) and (3.17), hence, to get (3.7) it is sufficient to prove that

\[
b_n^{-1}\sum_{j=1}^{n}(Z_j - EZ_j) \to 0 \quad \text{a.s.} \quad \tag{3.18}
\]

Recall that \( x^2/\varphi(x) \) is nondecreasing for both cases (a) and (b), as in the proof of Corollary 3.1, we have

\[
\frac{X_j^2}{b_j^2} \leq \frac{\varphi(X_j)}{\varphi(b_j)} \quad \text{for } |X_j| \leq b_j. \tag{3.19}
\]

Hence

\[
EZ_j^2 = EX_j^2I(|X_j| \leq b_j) + b_j^2P(|X_j| > b_j)
\]

\[
= b_j^2E(X_j^2/b_j^2)I(|X_j| \leq b_j) + b_j^2P(|X_j| > b_j)
\]

\[
\leq b_j^2E\varphi(X_j)I(|X_j| \leq b_j)/\varphi(b_j) + b_j^2P(|X_j| > b_j). \tag{3.20}
\]

Thus, using Lemma 3.2 and noting that (3.3), (3.5) and (3.6), we obtain that

\[
\sum_{n=1}^{\infty}n^{-1}P\left(\max_{1 \leq k \leq n}\left|\sum_{j=1}^{k}(Z_j - EZ_j)\right| > b_n\right)
\]

\[
\leq e^{-2}\sum_{n=1}^{\infty}n^{-1}b_n^{-2}E\max_{1 \leq k \leq n}\left|\sum_{j=1}^{k}(Z_j - EZ_j)\right|^2
\]

\[
\leq C\sum_{n=1}^{\infty}b_n^{-2}\left(\max_{1 \leq j \leq n}EZ_j^2 + 1\right)
\]

\[
\leq C\sum_{n=1}^{\infty}b_n^{-2}\left\{\max_{1 \leq j \leq n}b_j^2[E\varphi(X_j)I(|X_j| \leq b_j)/\varphi(b_j) + P(|X_j| > b_j)] + 1\right\}
\]

\[
< \infty. \tag{3.21}
\]

So we have got the desired result (3.18) by Theorem 2.1. \( \square \)

4. Strong law of large numbers for NA sequence

Here we will give the corresponding results for negatively associated random variables.
Definition 4.1. The random variables $X_1, X_2, \ldots, X_n$ are said to be negatively associated if for every $n$ and every pair of disjoint subsets $A_1, A_2$ of $\{1, 2, \ldots, n\}$,
\[ \text{Cov}(f_1(X_i : i \in A_1), f_2(X_j : j \in A_2)) \leq 0, \]
whenever $f_1$ and $f_2$ are coordinatewise increasing and this covariance exists.

Lemma 4.1 (Yang, 2000, Corollary 3, Yang, 2001; Shao, 2000). Let $X_1, X_2, \ldots$ be negatively associated random variables with zero means and $E|X_j|^r < \infty$, where $r > 1$. Then there exists a positive constant $C$, which does not depend on $n$, such that
\[ E \max_{1 \leq k \leq n} |S_n|^r \leq C \sum_{j=1}^n E|X_j|^r \quad \text{as } 1 < r \leq 2 \]  
(4.1)  
and
\[ E \max_{1 \leq k \leq n} |S_n|^r \leq C \left\{ \sum_{j=1}^n E|X_j|^r + \left( \sum_{j=1}^n E X_j^2 \right)^{r/2} \right\} \quad \text{as } r > 2. \]
(4.2)

Theorem 4.1. Let $\{X_j : j \geq 1\}$ be a sequence of negatively associated random variables. Let $\varphi : R \to R^+$ be an even and nondecreasing on $[0, \infty)$ function with $\lim_{x \to \infty} \varphi(x) = \infty$, and such that
\[(a) \quad \varphi(x)/x \searrow, \quad \text{or} \quad (b) \quad \varphi(x)/x^2 \nearrow, \quad x \to \infty \text{ and } EX_j = 0.\]

Assume further that $b_1, b_2, \ldots$ is a nondecreasing sequence of positive numbers satisfying
\[ 1 \leq b_{2n}/b_n \leq c < \infty \quad \text{for all } n \geq 1, \]  
(4.3)
\[ \sum_{j=1}^\infty \frac{E \varphi(X_j)}{\varphi(b_j)} < \infty, \]  
(4.4)
\[ \sum_{n=1}^\infty n^{-1}b_n^{-2} \sum_{j=1}^n b_j^2 P(|X_j| > b_j) < \infty, \]  
(4.5)
\[ \sum_{n=1}^\infty n^{-1}b_n^{-2} \sum_{j=1}^n b_j^2 \frac{E \varphi(X_j) \mathbb{I}(|X_j| \leq b_j)}{\varphi(b_j)} < \infty. \]  
(4.6)

Then
\[ S_n/b_n \to 0 \quad \text{a.s.} \]  
(4.7)

Proof. Along with the same line of proof of Theorem 3.1 but using Lemma 4.1 instead of Lemma 3.2. □

Taking $\varphi(x) = |x|^p$ and $b_n = (n \log n (\log \log n)^{\delta})^{1/p}$ in Theorem 4.1, we immediately obtain the following corollary.

Corollary 4.1. Assume that $\{X_j : j \geq 1\}$ is a sequence of negatively associated random variables with zero means and $\sup_{j \geq 1} E|X_j|^p < \infty$ for some $1 \leq p \leq 2$. Then for any $\delta > 1$,
\[ S_n/(n \log n (\log \log n)^{\delta})^{1/p} \to 0 \quad \text{a.s.} \]  
(4.8)

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References