Asymptotic ruin probabilities for proportional investment under interest force with dominatedly-varying-tailed claims

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ARTICLE INFO

Article history:
Received 29 January 2011
Accepted 24 June 2011
Available online 20 July 2011

AMS 2000 subject classification:
62E20
60G70

Keywords:
Ruin probabilities
Interest force
Dominatedly varying
Quasi-asymptotically independent

ABSTRACT

We study the asymptotic behavior of the ruin probabilities in the renewal risk model, in which the insurance company is allowed to invest a constant fraction of its wealth in a stock market which is described by a geometric Brownian motion and the remaining wealth in a bond with nonnegative interest force. We give the expression of the wealth process by the Itô formula, and finally we derive the asymptotic behavior of finite-time and infinite-time ruin probabilities in the presence of pairwise quasi-asymptotically independent claims with dominant varying tails for this model. In the particular case of compound Poisson model, explicit asymptotic expressions for the ruin probabilities are given with tails of regular variation, where the relation of the infinite-time ruin probability is the same as Gaier and Grandits (2004). For this case, we give some numerical results to assess the qualities of the asymptotic relations.

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1. Introduction and model

Recently, there has been a great attention on the problem of investment in the stock market for an insurer. This is due to the fact that the insurance company is allowed to invest its surplus in the stock market, described by a geometric Brownian motion. The classical works that are referred to are Asmussen and Taksar (1995), Browne (1995, 1997, 1999), Hipp and Plum (2000, 2003) and Paulsen and Gjessing (1997), and so on.

We model the risk process of an insurance company in the standard framework of the renewal risk model. In this model, \( \{X_i, i \geq 1\} \) is a sequence of nonindependent and nonnegative random variables with common distribution \( F \), modeling the size of the claim incurred by the insurer. And inter-occurrence times \( \{\theta_i, i \geq 1\} \) form another sequence of independent and identically distributed positive random variables independent of the sequence \( \{X_i, i \geq 1\} \). The arrival times of the successive claims \( T_n = \sum_{i=1}^n \theta_i \), constitute a renewal counting process

\[
N(t) = \sup\{n \geq 1 : T_n \leq t\}, \quad t \geq 0,
\]  

where by convention the supremum of the empty set is 0. Assume that the renewal process \( N(t) \) has a renewal function \( \lambda(t) = EN(t) \). For simplicity we denote the renewal function \( \lambda_t = \lambda(t) \). The surplus process \( U(t) \) is then expressed as

\[
U(t) = u + ct - \sum_{k=1}^{N(t)} X_k, \quad t \geq 0,
\]

where \( u \) is the initial capital of the insurance company, \( c > 0 \) is the constant premium rate, and \( \sum_{k=1}^{\infty} X_k = 0 \) by convention.
This classical risk model does not consider investment of the surplus. Assume that the insurance company not only invests in a bond with interest force \( r \) but also invests in a stock. We introduce a stock price process \( S(t) \), which is modeled by a geometric Brownian motion, i.e.,
\[
dS(t) = aS(t)dt + bS(t)dW(t),
\]
where \( a, b \in \mathbb{R} \) are fixed constants and \( W(t) \) is a standard Brownian motion independent of the surplus process \( U(t) \).

From an economic point of view, it seems that ‘rich’ companies should invest more in the stock than ‘poor’ companies. In many countries, insurance companies invest a constant fraction of the surplus in the risky assets at each point of time, and the remaining part in the bond. If at time \( t \), the insurer has wealth \( Y(t) \), and invests an amount \( ky(t) \) in stocks and \( (1-k)y(t) \) in bonds (with interest force \( r \)), the stochastic differential equation for this wealth \( Y(t) \) is
\[
dY(t) = dU(t) + ky(t)(adt + bdW(t)) + (1-k)rY(t-\)dt, \quad t \geq 0.
\]

Applying Itô’s formula, we obtain the expression of the wealth process \( Y(t) \):
\[
Y(t) = \exp\left(\left(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2\right)t + \tilde{\beta}W(t)\right) \cdot \left(u + c \cdot \int_0^t e^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)s - \tilde{\beta}W(s)} ds - \sum_{k=1}^{N(t)} X_k e^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)s - \tilde{\beta}W(s)}\right)
\]
where \( \tilde{\alpha} := (a-r)k + r \) and \( \tilde{\beta} := bk \). We aim to study the finite-time and infinite-time ruin probabilities of the risk model (3). As usual, the finite-time ruin probability \( \psi(u, T) \) is
\[
\psi(u, T) = P(Y(t) < 0 \text{ for some } 0 < t \leq T);
\]
and the infinite-time ruin probability is
\[
\psi(u) = P(Y(t) < 0 \text{ for some } 0 < t < \infty).
\]

If the insurance company is allowed to invest a constant fraction of its surplus in the stock market and the remaining reserve in a bond with nonnegative interest, there is also quite a lot of literature for the case, see Frapollo, Kabanov, and Pergamenshchikov (2002), Gaier and Grandits (2004), Kalashnikov and Norberg (2002), Nyrhinen (2001), Paulsen (2002), Wang and Wu (2001), and Wei (2009), etc. Most of them obtain similar results for infinite-time ruin probability by deriving the Hamilton–Jacobi–Bellman equation while in many cases it is very difficult to solve. For claims with tails of regular variation, Gaier and Grandits (2004) obtained analogous results of the infinite-time ruin probabilities for proportional investment as Proposition 4.1 of Paulsen (2002) by using different methods; they used an inductive method while Paulsen (2002) used a Laplace transform. All of them use the Poisson process to represent the number of claims up to time \( t \) and cannot obtain the finite-time ruin probability. For claims with tails of extended regular variation class, Wei (2009) investigated the infinite-time ruin probability for the proportional investment with independent and identically distributed claim sizes under zero interest force.

In this paper, when the number of claims is a renewal process and the claim sizes are pairwise quasi-asymptotically independent, we give both the finite-time and infinite-time ruin probabilities for proportional investment in the presence of dominated-varying-tailed claims sizes. For the particular case of consistently-varying-tailed claims sizes, our results extended Theorem 3.1 of Wei (2009). For another particular case of Poisson process, we obtain the same result as Corollary 14 of Gaier and Grandits (2004) (Proposition 4.1 of Paulsen, 2002) immediately from our theorem, and we also give some numerical results to assess the qualities of this asymptotic relations.

The rest of our paper is constructed as follows. Section 2 introduces preliminaries and presents our main results and our numerical results, Section 3 gives some necessary lemmas and Section 4 provides the proofs of the theorems.

2. Preliminaries and main results

The usual assumption about the distribution function (d.f.) \( F \) of \( X \) is that \( F \) is heavy-tailed. We say a random variable \( X \) or a d.f. \( F \) belongs to the class of heavy-tailed distributions, if \( E e^{\alpha X} = \infty \), for all \( \alpha > 0 \). Here we review some important classes of heavy-tailed distributions.

We say that a d.f. \( F \) on \( [0, \infty) \) is long-tailed, denoted as \( F \in \mathcal{L} \), if for any \( y > 0 \), we have
\[
\lim_{x \to \infty} \frac{F(x+y)}{F(x)} = 1.
\]
We say that a d.f. \( F \) belongs to the extended-regularly-varying-tailed class, denoted as \( F \in \mathcal{ERV}(-\alpha, -\beta) \), if there exist some \( 0 < \alpha \leq \beta < \infty \) such that for any \( y > 1 \), we have
\[
y^{-\beta} \leq \liminf_{x \to \infty} \frac{F(xy)}{F(x)} \leq \limsup_{x \to \infty} \frac{F(xy)}{F(x)} \leq y^{-\alpha}.
\]
If \( \alpha = \beta \), we say that \( F \) belongs to the regularly-varying-tailed class, denoted \( F \in \mathcal{R}_{-\alpha}, 0 < \alpha < \infty \).
We say that a d.f. $F$ belongs to the dominatedly-varying-tailed class, denoted as $F \in \mathcal{D}$, if for any $y > 0$, we have
\[
\limsup_{x \to \infty} \frac{F(xy)}{F(x)} < \infty.
\] (9)

We say that a d.f. $F$ belongs to the consistently-varying-tailed class, denoted as $F \in \mathcal{C}$, if
\[
\lim_{\mu \uparrow 1} \limsup_{x \to \infty} \frac{F(\mu x)}{F(x)} = 1, \quad \text{or} \quad \lim_{\mu \downarrow 1} \liminf_{x \to \infty} \frac{F(\mu x)}{F(x)} = 1.
\] (10)

The following relationships are well known:
\[
\mathcal{R} \subset \mathcal{ERV} \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{L}.
\]
For more details about the classes of heavy-tailed distributions, please refer to Bingham, Goldie, and Teugels (1987) and Embrechts, Klüppelberg, and Mikosch (1997). For some examples to illustrate the relationships between those classes, see Cai and Tang (2004) and Cline and Samorodnitsky (1994) and etc.

Now we introduce another index for the distribution $F$. We adopt the definitions in Tang and Tsitsiashvili (2003):
\[
\lambda_F^+= - \lim_{y \to \infty} \frac{\log F_+(y)}{\log y}, \quad \lambda_F^- = - \lim_{y \to \infty} \frac{\log F_-(y)}{\log y},
\]
where $F_+(y) = \lim \inf_{x \to \infty} F(xy)/F(x)$ and $F_-(y) = \lim \sup_{x \to \infty} F(xy)/F(x)$. These indices were introduced in Bingham et al. (1987) first, and later Cline and Samorodnitsky (1994) restudied them. But for our purpose, the way in Tang and Tsitsiashvili (2003) is more convenient. It is easy to see that $0 \leq \lambda_F^- \leq \lambda_F^+ \leq \infty$. If $F \in \mathcal{ERV}(-\alpha, -\beta)$, then $\alpha \leq \lambda_F^- \leq \lambda_F^+ \leq \beta$. $F \in \mathcal{D}$ if and only if $\lambda_F^+ < \infty$.

Now we introduce another index for the distribution $F$ as in Yi, Chen, and Su (2011) or Yang and Wang (2010): $L_F = \lim_{y \to 1} \tilde{F}_*(y)$. By the definition of the class $\mathcal{C}$, it is easy to see $L_F = 1$ for $F \in \mathcal{C}$.

Now let us introduce the definition of the dependence structure of random variables we will adopt: quasi-asymptotic independence. This is defined in Chen and Yuen (2009).

**Definition 2.1 (Quasi-Asymptotic Independence).** Two nonnegative random variables $X_1$ and $X_2$, with distributions $F_1$ and $F_2$ respectively, are said to be quasi-asymptotically independent if
\[
\lim_{x \to \infty} \frac{\Pr(X_1 > x, X_2 > x)}{\Pr(X_1 > x) + \Pr(X_2 > x)} = 0
\] (11)
holds.

In fact random variables $X_1$ and $X_2$ with identical distributions are called to be asymptotically independent or upper-tail independent if the relation (11) holds. See, for example, Zhanget al. (2009).

We close this section by explaining some symbols which will be used later. We will use $\preceq, \succeq,$ and $\sim$ to connect two functions, say $f_1(x)$ and $f_2(x)$, as follows: $f_1(x) \preceq f_2(x)$ when $\lim\sup_{x \to \infty} \frac{f_1(x)}{f_2(x)} \leq 1$; $f_1(x) \succeq f_2(x)$ when $\lim\inf_{x \to \infty} \frac{f_1(x)}{f_2(x)} \geq 1$; $f_1(x) \sim f_2(x)$ when $\lim\inf_{x \to \infty} \frac{f_1(x)}{f_2(x)} = 1$. Hereafter, all limit relationships are for $u \to \infty$ unless stated otherwise.

Thus the main result of the paper is given as follows.

**Theorem 2.1.** Consider the model (3) introduced in Section 1. Let claim sizes $(X_n, n \geq 1)$ be a sequence of pairwise quasi-asymptotically independent and nonnegative random variables with common distribution $F \in \mathcal{D}$, $0 < \lambda_F^- \leq \lambda_F^+ < 2\alpha/\beta - 1$. We have
\[
L_F \int_0^{\infty} \int_0^{\infty} F(uy)\lambda(y, \tilde{\mu} t, \tilde{\mu}^2 t)dx dt \lesssim \psi(u, T) \lesssim L_F^{-2} \int_0^{\infty} \int_0^{\infty} F(uy)\lambda(y, \tilde{\mu} t, \tilde{\mu}^2 t)dx dt
\]
where $\tilde{\mu} = \tilde{\alpha} - \tilde{\beta}^2/2$ and $\lambda(y, a, b^2)$ denotes the lognormal distribution with parameters $a$ and $b^2$.

According to Theorem 2.1 and Corollary 3.4 of Chen and Yuen (2009), after simple calculating we can obtain the following corollary.

**Corollary 2.1.** Consider the model (3) introduced in Section 1. Let claim sizes $(X_n, n \geq 1)$ be a sequence of pairwise quasi-asymptotically independent and nonnegative random variables with common distribution $F$. Then
\[
(1) \quad \text{If } F \in \mathcal{C}, 0 < \lambda_F^- \leq \lambda_F^+ < 2\alpha/\beta^2 - 1,
\]
\[
\psi(u, T) \sim \int_0^{\infty} \int_0^{\infty} F(uy)\lambda(y, (\tilde{\alpha} - \tilde{\beta}^2/2)t, \tilde{\beta}^2 t)dx dt
\]
(2) If \( F \in \mathcal{R}_{-\rho} \) for some \( 0 < \rho < 2\tilde{\alpha}/\tilde{\beta}^2 - 1 \), if \( N(t) \) is a Poisson process with intensity \( \lambda \), we have
\[
\psi(u, T) \sim \tilde{F}(u) \frac{\lambda (1 - e^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)\rho T + \frac{1}{2}\tilde{\beta}^2\rho^2 T})}{(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2) \rho - \frac{1}{2}\tilde{\beta}^2 \rho^2}.
\]

Next we consider the infinite-ruin probability of the model (3), we also assume that \( \psi_T^+ < 2\tilde{\alpha}/\tilde{\beta}^2 - 1 \) as in Theorem 2.1. The assumption shows that the volatility should be dominated by the drift, otherwise, large volatility will result in the bankruptcy with probability one, for details see Frovola et al. (2002).

**Theorem 2.2.** Assume the conditions of Theorem 2.1. If \( F \in \mathcal{D} \) for \( 0 < J_F^- \leq J_F^+ < 2\tilde{\alpha}/\tilde{\beta}^2 - 1 \), we have
\[
L_F \int_0^\infty \int_0^\infty \tilde{F}(uy) dLN(y, \tilde{\mu}, \tilde{\beta}^2 t) d\lambda_t \leq \psi(u) \leq L_F^{-2} \int_0^\infty \int_0^\infty \tilde{F}(uy) dLN(y, \tilde{\mu}, \tilde{\beta}^2 t) d\lambda_t
\]
where \( \tilde{\mu} = \tilde{\alpha} - \tilde{\beta}^2 / 2 \).

**Remark 2.1.** When \( F \in \mathcal{C} \) with \( 0 < J_F^- \leq J_F^+ < 2\tilde{\alpha}/\tilde{\beta}^2 - 1 \), that is \( L_F = 1 \), then the relation
\[
\psi(u) \sim \int_0^\infty \int_0^\infty \tilde{F}(uy) dLN(y, (\tilde{\alpha} - \tilde{\beta}^2 / 2)t, \tilde{\beta}^2 t) d\lambda_t
\]
holds. For any \( 0 < \alpha \leq \beta < \infty \), Cline and Samorodnitsky (1994) show that \( \mathcal{C} \) with \( J_F^- > 0 \) is strictly larger than \( \mathcal{C}_r(-\alpha, -\beta) \), thus Theorem 2.2 extends Theorem 3.1 of Wei (2009) (where \( r = 0 \)).

By the one-dimensional version of Theorem 2.1 of Resnick and Willekens (1991) we have the following corollary after simple calculating.

**Corollary 2.2.** Consider the model introduced in Section 1, \( \{X_n, n \geq 1\} \) are independent r.v.s with common d.f. \( F \in \mathcal{R}_{-\rho} \) for some \( \rho > 0 \), for \( 2\tilde{\alpha}/\tilde{\beta}^2 - 1 > \rho \) and \( N(t) \) is a Poisson process with intensity \( \lambda \), we have
\[
\psi(u) \sim \tilde{F}(u) \frac{\lambda}{(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2) \rho - \frac{1}{2}\tilde{\beta}^2 \rho^2}.
\]

**Remark 2.2.** This Corollary 2.2 is just the same as Corollary 14 of Gaier and Grandits (2004). If \( b = 0 \) (i.e., \( \beta = 0 \)) and \( k = 0 \), the model becomes a risk model with constant interest rate in Tang (2005).

To assess the qualities of the asymptotic relations in Corollaries 2.1 and 2.2 by simulation, we assume that premium rate \( c = 20 \) and intensity of the Poisson claim arrival process \( \lambda = 0.8 \). Table 1 gives some numerical results of Corollaries 2.1 and 2.2 for Pareto claims, the tail of which has the form \( F(u) = (1 + v/u)^{-\eta}, v, \eta, u > 0 \). We take the parameters above with values \( v = 0.2, \eta = 1.1 \). And assume \( a = 3.825, b = 1.25, r = 0.05, k = 0.4 \).

3. Some lemmas

In order to prove our theorems, we need some lemmas. Firstly, we show some properties of class \( \mathcal{D} \) which can be found in Proposition 2.2.1 of Bingham et al. (1987), Lemma 3.5 of Tang and Tsitsiashvili (2003) and Theorem 3.3 of Cline and Samorodnitsky (1994).

**Lemma 3.1.** For a r.v. \( X \) on \([0, \infty)\) with a distribution \( F \),
(1) \( F \in \mathcal{D} \iff L_F > 0 \).
(2) \( F \in \mathcal{D} \), then \( x^p = o(F(x)) \) for all \( p > J_F^+ \).
(3) If \( F \in \mathcal{D} \) and \( EY^p \) is finite for some \( p > J_F^+ \), then
\[
0 \leq E[F^+(1/Y)] \leq \lim \inf \frac{P(XY > t)}{F(t)} \leq \lim \sup \frac{P(XY > t)}{F(t)} < E[F^+(1/Y)] < \infty.
\]

The following two lemmas give some properties of the integral of exponential Brownian motion. Lemma 3.2 shows that there exists any finite moment for the integral on \([0, t] \) of exponential Brownian motion, the proof of which can be found in Yor (1992). Lemma 3.3 gives the distribution of the integral \([0, \infty) \) of exponential Brownian motion, and we can find its proof in Dufresne (1990).
4.1. Proof of Theorem 2.1

Our proof below is motivated by an idea of Tang (2005).

\[ \widetilde{Y}(t) := e^{-\left(\widetilde{\alpha} - \frac{1}{2} \widetilde{\beta}^2\right)t - \widetilde{\beta}W(t)} \]

\[ = u + c \cdot \int_0^t e^{-\left(\widetilde{\alpha} - \frac{1}{2} \widetilde{\beta}^2\right)s - \widetilde{\beta}W(s)} ds - \sum_{k=1}^{N(t)} X_k e^{-\left(\widetilde{\alpha} - \frac{1}{2} \widetilde{\beta}^2\right)T_k - \widetilde{\beta}W(T_k)}. \]

Denote

\[ C(t) := c \cdot \int_0^t e^{-\left(\widetilde{\alpha} - \frac{1}{2} \widetilde{\beta}^2\right)s - \widetilde{\beta}W(s)} ds \]

and

\[ \widetilde{X}(t) := \sum_{k=1}^{\infty} X_k e^{-\left(\widetilde{\alpha} - \frac{1}{2} \widetilde{\beta}^2\right)T_k - \widetilde{\beta}W(T_k)} I(T_k \leq t), \]

\begin{table}[h]
\centering
\caption{Simulated versus asymptotic values of the ruin probabilities for \( T = 10 \) and \( T = 100 \).}
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
\( u \) & \multicolumn{3}{c|}{\( T = 10 \)} & \multicolumn{3}{c|}{\( T = 100 \)} \\
\hline
& Simulated & Asymptotic & 1 - Simulated & Asymptotic & 1 - Simulated & Asymptotic & 1 - Simulated \\
\hline
100 & 0.001203 & 0.001732 & 0.305 & 0.001296 & 0.001736 & 0.253 \\
300 & 0.000424 & 0.000514 & 0.175 & 0.000457 & 0.000519 & 0.120 \\
500 & 0.000239 & 0.000290 & 0.176 & 0.000268 & 0.000296 & 0.095 \\
800 & 0.000161 & 0.000170 & 0.053 & 0.000164 & 0.000176 & 0.068 \\
1000 & 0.000131 & 0.000137 & 0.044 & 0.000131 & 0.000138 & 0.051 \\
\hline
\end{tabular}
\end{table}
where $I(A)$ denotes the indicator function of an event $A$. Note $\lambda_1 = \sum_{k=1}^{\infty} P(T_k \leq t)$, we know for any $\rho > 0$

$$\sum_{k=1}^{\infty} E e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho T_k - \rho \tilde{\beta} W(T_k) \right)} I(T_k \leq T) = \sum_{k=1}^{\infty} \int_0^T E e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho t - \rho \tilde{\beta} W(t) \right)} dP(T_k \leq t)$$

$$= \sum_{k=1}^{\infty} \int_0^T e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho t + \frac{1}{2} \rho^2 \tilde{\beta}^2 t \right)} dP(T_k \leq t)$$

$$= \int_0^T e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho t + \frac{1}{2} \rho^2 \tilde{\beta}^2 t \right)} d\lambda_1. \quad (16)$$

By the relation (16), the condition of Lemma 3.4 is satisfied for $0 < J_F^+ < 1$. While for the case $1 \leq J_F^+ < 2\alpha / \beta^2 - 1$, choosing $\delta > 0$ such that $p_2 = J_F^+ + \delta < 2\alpha / \beta^2 - 1$, then for any $0 < p \leq p_2$ and by Brownian motion’s independent and stationary increments,

$$\sum_{k=1}^{\infty} (E e^{-p \left((\alpha - \frac{1}{2} \beta^2) T_k - \rho \tilde{\beta} W(T_k) \right)})^{\frac{1}{p_2}} \leq \sum_{k=1}^{\infty} (E e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho T_k - \rho \tilde{\beta} W(T_k) \right)})^{\frac{1}{p_2}}$$

$$= \sum_{k=1}^{\infty} (E e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho (T_k - T_{k-1}) - \rho \tilde{\beta} (W(T_k) - W(T_{k-1})) \right)})^{\frac{1}{p_2}}$$

$$\times E e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho T_{k-1} - \rho \tilde{\beta} W(T_{k-1}) \right)}^{\frac{1}{p_2}}$$

$$= \cdots = \sum_{k=1}^{\infty} \left( \sum_{j=1}^{k} \left( E e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho T_j - \rho \tilde{\beta} W(T_j) \right)} \right)^{\frac{1}{p_2}} \right)$$

$$= \sum_{k=1}^{\infty} (E e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho T_k - \rho \tilde{\beta} W(T_k) \right)})^{\frac{k}{p_2}} < \infty, \quad (17)$$

where inter-occurrence times $\theta_j = T_j - T_{j-1}$, $j \geq 1$. The last relation follows by $E e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho \theta_1 - \rho \tilde{\beta} W(\theta_1) \right)} = E e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho \theta_1 + \frac{1}{2} \rho^2 \tilde{\beta}^2 \theta_1 \right)} < 1$. Thus the condition for the case $J_F^+ > 1$ of Lemma 3.4 is also satisfied. It is clear that we can rewrite the finite ruin probability

$$\psi(u, T) = P(\tilde{Y}(t) < 0 \text{ for some } 0 \leq t \leq T)$$

and that

$$u - \tilde{X}(T) \leq \tilde{Y}(t) \leq u + C(T) - \tilde{X}(t). \quad (18)$$

Using the first inequality of (18) and Lemma 3.4, we have

$$\psi(u, T) \leq P(\tilde{X}(T) > u) = P\left( \sum_{k=1}^{\infty} X_k e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho T_k - \rho \tilde{\beta} W(T_k) \right)} I(T_k \leq T) > u \right)$$

$$\leq L_F^{-2} \sum_{k=1}^{\infty} \int_0^T P(X_k e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho t - \rho \tilde{\beta} W(t) \right)} > u) dP(T_k \leq t)$$

$$= L_F^{-2} \sum_{k=1}^{\infty} \int_0^T P(X_k e^{-\left((\alpha - \frac{1}{2} \beta^2) \rho t - \rho \tilde{\beta} W(t) \right)} > u) dP(T_k \leq t)$$

$$= L_F^{-2} \sum_{k=1}^{\infty} \int_0^T \int_0^\infty \tilde{F}(uy) d\lambda_1(y, (\alpha - \frac{1}{2} \beta^2) t, \tilde{\beta}^2 t) dP(T_k \leq t)$$

$$= L_F^{-2} \int_0^T \int_0^\infty \tilde{F}(uy) d\lambda_1(y, (\alpha - \frac{1}{2} \beta^2) t, \tilde{\beta}^2 t) d\lambda_1. \quad (19)$$

On the other hand, we consider the lower bound. By Lemma 3.2 we can obtain that $C(T)$ has finite moments of all orders, then $\lim_{x \to \infty} x^\rho P(C(T) > x) = 0$, $\rho > 0$. While by Lemma 3.1 $\lim_{x \to \infty} x^\rho \tilde{F}(x) = +\infty$ for any $\rho > J_F^+$. So

$$\lim_{x \to \infty} \frac{P(C(T) > x)}{\tilde{F}(x)} = \lim_{x \to \infty} \frac{x^\rho P(C(T) > x)}{x^\rho \tilde{F}(x)} = 0, \quad \rho > J_F^+. \quad (20)$$
By the second inequality of (18), for any $0 < l < 1$ we derive
\[
\psi(u, T) \geq P(\widetilde{X}(t) > u + C(T) \text{ for some } 0 \leq t \leq T) = P(\widetilde{X}(T) > u + C(T)) \geq P(\widetilde{X}(T) > (1 + l)u) - P(C(T) > lu).
\]

By Lemma 3.4, we have
\[
P(\widetilde{X}(T) > (1 + l)u) = P\left(\sum_{k=1}^{\infty} X_k e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)T_k - \bar{\beta}W(T_k)} | (T_k \leq T) > (1 + l)u\right)
\geq \sum_{k=1}^{\infty} P(X_k e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)T_k - \bar{\beta}W(T_k)} | (T_k \leq T) > (1 + l)u)
\geq \sum_{k=1}^{\infty} \int_{0}^{T} P(X_k e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)t - \bar{\beta}W(t)} > (1 + l)u) dP(T_k \leq t)
\geq \int_{0}^{T} P(X \exp\{-(\bar{\alpha} - \bar{\beta}^2/2)t - \bar{\beta}W(t)\} > (1 + l)u) d\lambda_t.
\] (21)

We continue to derive the lower bound of (22), for a fixed $0 < \epsilon < 1$,
\[
\int_{0}^{T} P(X \exp\{-(\bar{\alpha} - \bar{\beta}^2/2)t - \bar{\beta}W(t)\} > (1 + l)u) d\lambda_t
\geq \int_{0}^{T} P(X_k e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)T_k - \bar{\beta}W(T_k)} > (1 + l)u, \ e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)T_k - \bar{\beta}W(T_k)} \leq u^{1-\epsilon}) d\lambda_t
= \int_{0}^{T} \int_{0}^{u^{1-\epsilon}} F((1 + l)u/y) dLN(y, -(\bar{\alpha} - \bar{\beta}^2/2)t, \bar{\beta}^2t) d\lambda_t
\geq F_s(1 + l) \int_{0}^{T} \int_{0}^{u^{1-\epsilon}} F(u/y) dLN(y, -(\bar{\alpha} - \bar{\beta}^2/2)t, \bar{\beta}^2t) d\lambda_t
= F_s(1 + l) \int_{0}^{T} \left[ \int_{0}^{\infty} - \int_{u^{1-\epsilon}}^{\infty} \right] F(u/y) dLN(y, -(\bar{\alpha} - \bar{\beta}^2/2)t, \bar{\beta}^2t) d\lambda_t
\geq F_s(1 + l) \left[ \int_{0}^{T} P(X e^{-(\bar{\alpha} - \bar{\beta}^2/2)t - \bar{\beta}W(t)} > u) d\lambda_t - \int_{0}^{T} P(e^{-(\bar{\alpha} - \bar{\beta}^2/2)t - \bar{\beta}W(t)} > u^{1-\epsilon}) d\lambda_t \right].
\] (23)

Next we shall prove that the second integral of (23) is negligible asymptotically compared with the first integral of (23). Choosing $h$ such that $h(1 - \epsilon) > \int_{0}^{T} F_s(e^{(\bar{\alpha} - \bar{\beta}^2/2)t + \bar{\beta}W(t)}) d\lambda_t$,
\[
\int_{0}^{T} e^{-h(1 - \epsilon)} \int_{0}^{T} E\{e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)t - \bar{\beta}W(t)} > u\} d\lambda_t
\leq \int_{0}^{T} \frac{E e^{-h(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)t - \bar{\beta}W(t)}}{u^{h(1 - \epsilon)}} d\lambda_t
= u^{-h(1 - \epsilon)} \int_{0}^{T} \exp \left\{ - \left( h(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2) - \frac{1}{2} \bar{\beta}^2 h^2 \right) t \right\} d\lambda_t.
\] (24)

On the other hand,
\[
\int_{0}^{T} P(X e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)t - \bar{\beta}W(t)} > u) d\lambda_t = \int_{0}^{T} P(X e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)t - \bar{\beta}W(t)} > u) F(u) d\lambda_t
\geq \int_{0}^{T} e^{\int_{0}^{T} F_s(e^{\frac{1}{2}\bar{\beta}^2 t + \bar{\beta}W(t)})} d\lambda_t.
\] (25)

the last inequality holds by Lemma 3.1. Combining (24) and (25), we have by Lemma 3.1(2),
\[
\lim_{u \to \infty} \frac{\int_{0}^{T} P(X e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)t - \bar{\beta}W(t)} > u^{1-\epsilon}) d\lambda_t}{\int_{0}^{T} P(X e^{-(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2)t - \bar{\beta}W(t)} > u) d\lambda_t} \leq \lim_{u \to \infty} \frac{u^{-h(1 - \epsilon)} \int_{0}^{T} \exp \left\{ - (h(\bar{\alpha} - \frac{1}{2}\bar{\beta}^2) - \frac{1}{2} \bar{\beta}^2 h^2) t \right\} d\lambda_t}{\int_{0}^{T} \int_{0}^{T} \int_{0}^{T} e^{\int_{0}^{T} F_s(e^{\frac{1}{2}\bar{\beta}^2 t + \bar{\beta}W(t)})} d\lambda_t} = 0.
\] (26)
Therefore combining this with (22) and (23), we have
\[
\lim_{l \to 0} P(\tilde{X}(T) > (1 + l)u) = \lim_{l \to 0} \int_0^T \int_0^{u_l} P(Xe^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)l - \tilde{\beta}W(t)} > u) d\lambda_t
\]
\[
= L_T \int_0^T \int_0^{\infty} \bar{F}(uy) dLN(y, (\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)t, \tilde{\beta}^2t) d\lambda_t.
\]
(27)

Then applying the Eq. (16) again, for any fixed number \(l > 0\),
\[
\lim_{u \to \infty} \frac{P(C(T) > lu)}{\bar{F}(lu)} = \lim_{u \to \infty} \frac{P(C(T) > lu)}{\bar{F}(lu)} \lim_{u \to \infty} \bar{F}(hu) \bar{F}(lu) \int_0^T \int_0^{\infty} P(Xe^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)l - \tilde{\beta}W(t)} > u) d\lambda_t
\]
\[
\leq \lim_{u \to \infty} \frac{P(C(T) > lu)}{\bar{F}(lu)} \lim_{u \to \infty} \bar{F}(hu) \bar{F}(lu) \int_0^T \int_0^{\infty} e^{(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)l + \tilde{\beta}W(t)} d\lambda_t
\]
\[
= 0.
\]
(28)
The last relation holds by the relation (20), the definition of class \(D\) and Lemma 3.1(3). Combining the above relation (28), (21) and (27), we have
\[
\psi(u, T) \geq L_T \int_0^T \int_0^{\infty} \bar{F}(uy) dLN(y, T, \tilde{\beta}^2t) d\lambda_t.
\]
(29)

Combining (19) with (29), we complete the proof.

4.2. Proof of Theorem 2.2

We prove this theorem by the similar way as the proof of Theorem 2.1.

Similarly we can also rewrite the ultimate ruin probability
\[
\psi(x) = P(\tilde{Y}(t) < 0 \text{ for some } t \geq 0)
\]
and note that
\[
u - \tilde{X} \leq \tilde{Y}(t) \leq u + C - \tilde{X}(t)
\]
where \(\tilde{X} = \tilde{X}(\infty) = \sum_{k=1}^{\infty} X_k e^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)T_k - \tilde{\beta}W(T_k)}\) and \(C = C(\infty) = c \int_0^{\infty} e^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)s - \tilde{\beta}W(s)} ds\). By the Lemma 3.3, we know
\[
\tilde{C} \sim IG\left(\frac{2(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)}{\tilde{\beta}^2}, \frac{2}{\tilde{\beta}^2}\right).
\]
For any \(2(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)/\tilde{\beta}^2 > \rho > 3^+_T\), then after simple calculations we have
\[
E\tilde{C}^\rho = \left(\frac{2}{\tilde{\beta}^2}\right)^\rho \Gamma\left(2\left(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2\right)/\tilde{\beta}^2\right) \Gamma\left(2\left(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2\right)/\tilde{\beta}^2 - \rho\right) < +\infty.
\]
Thus by Lemma 3.1 we have for any \(2(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)/\tilde{\beta}^2 > \rho > 3^+_T\),
\[
\lim_{u \to \infty} \frac{P(\tilde{C} > u)}{u^\rho \bar{F}(u)} = \lim_{u \to \infty} \frac{u^\rho P(\tilde{C} > u)}{u^\rho \bar{F}(u)} = 0.
\]
For \(2(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)/\tilde{\beta}^2 > \rho > 3^+_T\), inter-occurrence times \(\theta_j = T_j - T_{j-1}\),
\[
\sum_{k=1}^{\infty} e^{-(\alpha - \frac{1}{2}\beta^2)T_{k-1} - \beta W(T_{k-1})} = \sum_{k=1}^{\infty} e^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)T_{k-1} - \tilde{\beta}W(T_{k-1})} e^{-(\alpha - \frac{1}{2}\beta^2)T_{k-1} - \beta W(T_{k-1})}
\]
\[
= \cdots = \sum_{k=1}^{\infty} (e^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)T_{k-1} - \tilde{\beta}W(T_{k-1})}) k
\]
\[
= \frac{e^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)T_{k-1} - \tilde{\beta}W(T_{k-1})}}{1 - e^{-(\tilde{\alpha} - \frac{1}{2}\tilde{\beta}^2)T_{k-1} - \tilde{\beta}W(T_{k-1})}} = \frac{e^{-(\alpha - \frac{1}{2}\beta^2)T_{k-1} + \frac{1}{2}\beta^2T_{k-1}}}{1 - e^{-(\alpha - \frac{1}{2}\beta^2)T_{k-1} + \frac{1}{2}\beta^2T_{k-1}}} < \infty,
\]
(31)
The last equation is by the law of total expectation. Then the conditions of Lemma 3.4 hold. By Lemma 3.4 and using the same approach as Theorem 2.1, we can prove this result.
Acknowledgments

Chen’s work is supported by National Science Foundation of China (No.10801124, 70871104), and Zhang’s work is supported by National Science Foundation of China (No.10801123).

References


